Effects of demography and social groups on opinion dynamics

Mathias MJ Bellaiche[1,2] and David A Broniatowski[3]

[1] Department of Chemistry, University of Cambridge, Lensfield Road, Cambridge, CB2 1EW, United Kingdom
mmjb2@cam.ac.uk
[2] Laboratory of Chemical Physics, National Institute of Digestive and Diabetes and Kidney Diseases, National Institutes of Health, Bethesda, Maryland, 20892, United States America
mathias.bellaiche@nih.gov
[3] Department of Engineering Management and Systems Engineering, The George Washington University, Washington, District of Columbia, 20052, United States of America
broniatowski@email.gwu.edu

Abstract. We examine here the dynamics of opinion exchange before a democratic decision-making event by using an agent-based model. The agents belong to both demographic and social groups, and these memberships determine respectively voting behaviour and inter-agent exchange of opinions through a non-linear mechanism based on the theory of chemical reactions. Our model results in consistent extremist polarisation of opinions and local opinion clustering, and highlights the stochastic nature of outcome predictions. The project suggests that a combination of a non-linear opinion interaction mechanism and social group self-preference is enough to polarise a society and generate a very close decision outcome.

Keywords: Opinion dynamics · Consensus formation · Voting behaviour · Physical chemistry

1 Introduction

There has been a recent surge in popular backlash against globalisation and its policy consequences, as exhibited by such high-profile events as the British people's vote to leave the European Union or the American people's election of Donald Trump to the presidency. Both the Leave and Trump campaigns ran on divisive populist and isolationist ideals, and their successes highlight the effects of political division in a society (on both demographic and social levels) and the fragility of national or supranational organisations when subject to decisions made by a divided society.

For example in the Brexit referendum, those aged 18–24 were about 75% in favour of remaining, whilst those aged older than 65 were about 60% in favour of leaving. Furthermore, those whose highest certification was a General Certificate of Secondary Education (GSCE; equivalent to an American high school diploma) were about 65% in favour of leaving, in contrast to those with university degrees, who were about 70% in favour of remaining. Regional discrepancies also arose: the 'Golden Triangle' of London, Cambridge and Oxford, as well as Scotland, emerged as main Remain centres, whilst regions of the United Kingdom with lower population densities and levels of urbanisation were in general more in support of the Leave campaign [1, 2].

Beyond highlighting social divisions, Brexit also had serious economic and social effects - the strength of the British Pound Sterling has plummeted, whilst the number of hate crimes against minorities in the days following the referendum rose sharply. The US election had similar effects in terms of reported hate crimes, though economically the strength of the dollar has risen in response to Trump's promise of a protectionist investment in American infrastructure (having dire consequences for the economies of developing countries who borrow in the American currency). These events have had catalytic effects on similar movements in other countries: prior to his election, Trump lauded the Leave victory and painted himself as 'Mr. Brexit', whilst post-election, the French Front National presidential candidate Marine Le Pen's chief adviser chillingly proclaimed 'their world is collapsing; ours is beginning to be built.' Politically, these campaigns have ultimately highlighted the susceptibility to anti-globalist forces of institutions once regarded as impervious to populism, at both the national (such as the political party establishment) and supranational (such as the European Union (EU)) levels [3–8].

As such, the mechanisms by which these events transpired are critical to understanding so as to mitigate against the consequences of a deeply divided society and to protect the virtues of globalisation against demagoguery.

In this project, we construct an agent-based model (ABM) to investigate the effects of demography and social groups on inter-agent interaction and the evolution of a group's opinion dynamics. We show that demographic-based voting behaviour renders polling results difficult to interpret and that preferential intra-group interactions result in local opinion clustering and global opinion polarisation. Overall, we are able to recapitulate the observed properties of close vote counts, extreme views and local opinion clustering in social group space characteristic of recent democratic decision events.

2 Literature Review

The problem of opinion dynamics, consensus formation and polarisation has been examined by groups in the past. These models all look at how agents who possess a continuous 'opinion' attribute interact with each other.

Krause in 2000 investigated the situation of interacting experts, asking the question under what conditions a consensus was guaranteed. The work was mathematical, using stochastic linear algebra and Markov chains, and showed that if all agents took weighted averages of all other agents' opinions at each time step, consensus (i.e. arrival of all agents to the same opinion fixed point) was guaranteed for both geometric and power averaging. If, however, the model was recast in the so-called 'bounded confidence' scheme, such that each agent interacts only with other agents whose opinions fall within some cutoff of their own opinion, then consensus was no longer guaranteed and clustering of different opinions was possible [9].

Dittmer in 2001 then expanded on Krause's work and defined under which conditions the bounded-confidence model yields consensus, and found that a necessary and sufficient condition for consensus is that all the agents' opinions form an ε-chain. That is to say, if the agents are ordered based on the magnitudes of their opinion, then no two neighbours are further apart than the model cutoff for interaction ε [10].

Deffuant and coworkers have also tackled the problem, using agent-based modelling rather than theoretical maths. In 2000, they simulated the bounded-confidence model for initially non-spatially explicit agents and found that global opinion tended towards a single medium opinion for high interaction cutoff, whilst multimodal opinion distributions centred around non-extreme opinion values resulted from using small interaction cutoffs. They then applied the model to an explicit spatial environment, using a lattice model in which each agent is only allowed to interact with its four von Neumann neighbours. The results were similar to the spatially implicit case, and showed well-mixed medium opinions for large cutoffs and the emergence of clusters of off-centre opinions for small cutoffs [11].

In 2002 the group revisited the problem, introducing a new ingredient to the model – each agent was now allowed to possess an uncertainty in addition to its opinion, and at every interaction iteration agents exchanged both opinion and uncertainty information. Evolution of each agent's opinion and uncertainty due to interaction with other agents was linearly proportional to the overlap with the other agent's opinion, and inversely proportional to agent uncertainties. This model, for identical initial uncertainties, generated the bounded confidence model results of splitting into clusters with opinions slightly off-centre. Introducing explicit 'extremist' agents, though, allowed for extreme polarisation of opinion to either end of the opinion spectrum. In their work, an extremist was an agent with an opinion on either end of the opinion spectrum with very low uncertainty and hence very high influence on others. [12]

Taken together, this past work has uncovered crucial general trends in opinion dynamics, including under what conditions convergence to a single global opinion is guaranteed, and also ways in which extremism and polarisation can emerge from inter-agent interactions. Past work, however, has yet to investigate the interaction of heterogeneous agents, and especially how the properties of an agent can influence the ways in which it interacts with others.

3 Method

The `Mesa` module of Python was used to create an opinion dynamics ABM. This model works on an explicit square $L \times L$ toroidal grid whose coordinates represent 'social group' space.

Each grid space is occupied by an agent, who has two static properties (social group and demography) and two dynamic attributes (opinion and inducibility). The social group of an agent defines the agents' coordinate, and can be one of either 'a' or 'b'. Social group correlation is introduced into the model by arranging the grid so that it is formed of two immiscible phases of either group (i.e., there are two contiguous regions of space populated by agents of group 'a' or 'b'). The demography of an agent is not spatially dependent, and can be one of either 'A' or 'B'. The time-dependent properties of an agent i, opinion $o_i(t)$ and inducibility $\varepsilon_i(t)$ are analogous respectively to the opinion and uncertainty attributes in [12]. The opinion takes a value between 0 and 1, defining which way of the political spectrum its agent is leaning, whilst the inducibility takes a value between 0 and 1 and defines a range of opinions $o_i(t) \pm \varepsilon_i(t)$ to which the agent is willing to be exposed.

Data from Brexit voting statistics were used to initialise the model [13–15], where the two demographic groups were voters under 50 and those older than 49, whilst the two social groups were those without a university degree and those with. In this case, the opinion spectrum extremes correspond to a preference to Leave the EU ($o_i = 0.0$) or to Remain in the EU ($o_i = 1.0$) See Table I in Appendix A for the exact parameters used, but in short the grid was populated to reflect English or UK populations of each group according to recent data, making sure to place agents of the same group in contiguous spatial blocks. Each agent's opinion was initialised by drawing from a normal distribution centred about the mean of that agent's demographic and social group average opinion, as determined by survey results post-referendum, and with standard deviation 0.25. This value of Gaussian width was chosen as its probability density function would cover all possible opinions within 2 sigma given an agent with exactly no initial preference. Initial inducibilities were generated by sampling a skewed beta-distribution with $\alpha = 2$, $\beta = 5$. This inducibility distribution mirrors a situation in which most people are relatively open to discourse, but are nevertheless more likely to interact positively with other agents who have similar opinions.

At each timestep, each agent (using a random serial synchronicity schedule) finds another agent and 'discusses' by calculating opinion spectral overlap. Then every agent i, given the group identity of the interaction partner j and the spectral overlap between the two Ω_{ij} adopts the other agent's opinion with probability p_{ij}, to be described later. If the transition $o_i(t+1) = o_j(t)$ is successful, then nothing else happens to the agent, whilst if it is unsuccessful, the agent's inducibility is reduced according to $\varepsilon_i(t+1) = \max(0, \varepsilon_i(t) - 0.0025)$, to model the assumed fact that peoples' opinions get more entrenched over time if they are not subject to successful ideological challenge. This loops until all agents have 0 inducibility ($\varepsilon_i = 0\,\forall\, i$), such that everyone has definitely made up their minds about how they will vote.

The psuedocode for the model is shown in Algorithm 1, and the Python code is reproduced in Appendices C and D.

Algorithm 1 ABM Psuedocode

```
while at least one agent is undecided do
    for every agent i do:
        search for agent j with whom to interact
        determine overlap Ω_ij with agent j
    for every agent i do:
        calculate opinion conversion probability p_ij
        attempt opinion conversion
        if conversion successful then:
            continue
        else
            reduce inducibility
```

The way in which each agent finds an interaction partner is by sampling a 2D spherical distribution function that is exponentially distributed in the radial direction and uniformally randomly distributed in the polar direction - this means that each agent is equally likely to find another agent in any direction. See Appendix B for mathematical details, but because of the polar Jacobian, the effective radial distribution function becomes $P_r(r) = Nr\exp(-\gamma r)$, for N a normalisation constant and $r \in (0, L/2)$ (remembering that the periodic boundary conditions place a limit on the maximum physically meaningful inter-agent separation as $L/2$ for grid side length L). The parameter γ determines the maximum of the probability density function (simple optimisation of P_r shows that $r_{\max} = \gamma^{-1}$), so that small values of γ means agents are likely to speak with agents far away, whilst large values mean the converse. Verification steps were taken to ensure that the coded sampling function matched the analytical probability density function by running the sampling algorithm 1000 times, generating probability histograms of the sampling and comparing against the analytical curves. These results are shown for various values of γ in Fig. 1, and show good agreement between the numerical and analytical versions, demonstrating successful verification.

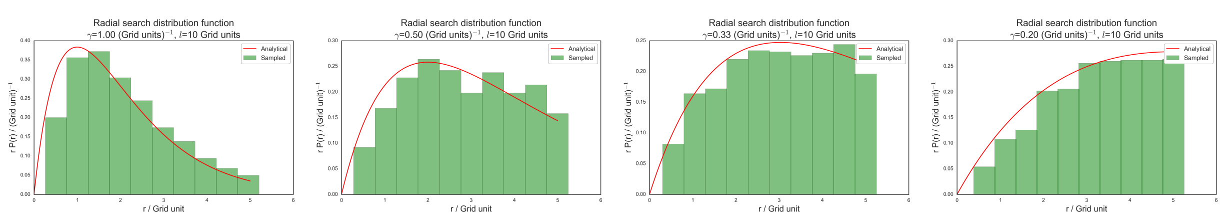


Fig. 1: Verification of radial sampling algorithm for $L = 10$ and (a) $\gamma = 1.0$, (b) $\gamma = 0.5$, (c) $\gamma = 0.33$ and (d) $\gamma = 0.20$ (grid units)$^{-1}$. Green histograms are the numerical sampling algorithm whilst red curves are the analytical distributions.

To determine the probability of opinion conversion p_{ij}, each agent i must consider two effects: the social group of its interaction partner j, and the spectral overlap Ω_{ij} of shared opinion space (Ω_{ij} is the length of overlap between the regions $[o_i(t) - \varepsilon_i(t), o_i(t) + \varepsilon_i(t)]$ and $[o_j(t) - \varepsilon_j(t), o_j(t) + \varepsilon_j(t)]$. The determination of p_{ij} from these ingredients is done by analogy to chemical reactions, in which the probability of a successful reactant encounter for activation free energy $\Delta_r G^\ddagger$ is Boltzmann-distributed as $p = \exp(-\Delta_r G^\ddagger/(RT))$, for R a constant and T the temperature [16]. In chemistry, the energy barrier effectively determines how hard it is to see a molecular configuration conducive to reaction, whilst the temperature determines how likely each molecule is to explore different configurations. Making the identification that the barrier $\Delta_r G^\ddagger$ is similar to group identity (in that belonging to the same social group predisposes agents to successful interactions, whereas belonging to different social groups does the opposite), and the temperature T is similar to opinion overlap (in that a stronger overlap means that an agent is more likely to be conducive to what its interaction partner has to say), generates the formulae for determining p_{ij} in this model (leaving out normalisation constants for aesthetics):

$$p_{ij} \sim \begin{cases} \exp(-(1-\sigma)/\Omega_{ij}) & \text{group } i = \text{group } j \\ \exp(-\sigma/\Omega_{ij}) & \text{group } i \neq \text{group } j, \end{cases} \tag{1}$$

for σ group self-preference, a model parameter.

Verification steps were taken to ensure that the coded p_{ij} matched Eq. 1 by running the calculation function as functions of Ω_{ij}. These results are shown for various values of σ in Fig. 2, and show good agreement between the numerical and expected function forms from considering the analytical versions, demonstrating successful verification.

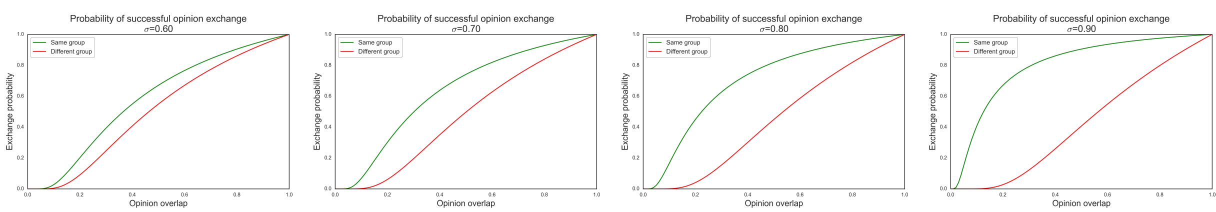


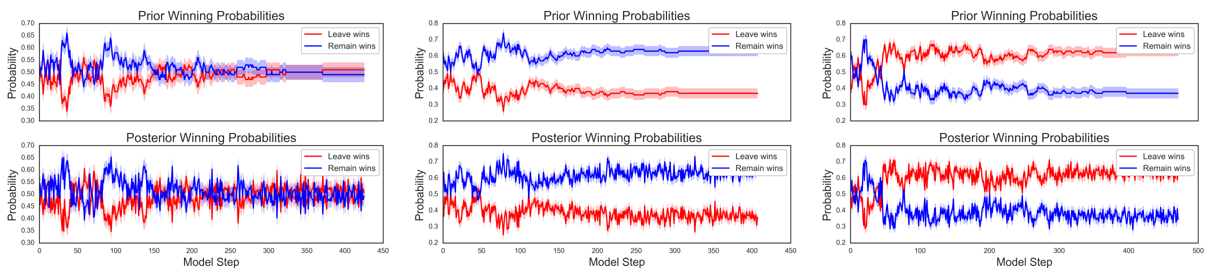
Fig. 2: Verification of interaction probability algorithm for (a) $\sigma = 0.6$, (b) $\sigma = 0.7$, (c) $\sigma = 0.8$ and (d) $\sigma = 0.9$. Green curves indicate interaction between members of the same social group, whilst red indicate interaction between members of different groups.

Finally, as the model runs, timecourses of the probability of either ideological side winning are generated by considering the number of agents with opinions less than, or greater than, 0.5 (those with opinion exactly 0.5 contribute half to either side's winning probability). This calculation is first done without regards to the effects that an agent's demography has on its likelihood to vote, generating a 'prior' timecourse. The calculation is then repeated by having each agent contribute to the vote with probability equal to its voting likelihood as determined by its demographic group, generating a 'posterior' timecourse.

4 Results

The model was run five times for each combination of parameters as detailed in Table I of Appendix A, for a total of 250 model runs. Given the large number of simulations, it is impossible to present all results, so a representative set and aggregate values will be shown instead. In these simulations, only the γ and σ parameters were changed, to investigate the effects respectively of changing search probabilities and group ideological cohesion on opinion dynamics.

Regarding prediction timecourses, it was observed that the same set of parameters would often yield simulations with strongly different results. For example, Fig. 3 shows the results of three simulations using the parameters $\gamma = 0.5$ (grid units)$^{-1}$ and $\sigma = 0.5$. These results underscore the inherent stochasticity of this process, the effects of which are diverging results from using the same parameters. Another point to make is the difference between the prior and posterior timecourses in each case - taking into account the randomness of any agent voting due to its demographic-based voting behaviour adds extra noise to the timecourses, making it difficult to make quantitative predictions about outcomes. Finally, notice that all timecourses converge on close results; it is important to remember that just because in some cases the probability of one side winning is higher than that of the other side does not guarantee victory. The probabilistic nature of this sort of simulated polling means that given, for example, a 60% chance of one side winning, if the referendum were to be run five times, the other side would win two of those times.



(a) Both sides have same probability. (b) Leave has higher probability. (c) Remain has higher probability.

Fig. 3: Winning probability timecourses for different simulations of the same parameter set. Blue curves indicate the probability of Remain winning, whilst red indicate the probability of Leave winning. Shades indicate ± 0.03 points, the standard margin of error of most polls.

Running this scan allows us to validate our model by examining its sensitivity to the parameters γ and σ by looking at global metrics. To do so, we examine the initial, final and percent change in overall average opinion and standard deviation of opinion as functions of γ and σ, with the results shown in Fig. 4. The average of opinions measures the global polarisation of the world, whilst the standard deviation of opinions measures the spread, or in some manner the local opinion polarisations. The results of the sensitivity analysis show that there is no strong effect of either γ or σ on the model results, save for perhaps a small positive trend of final opinion standard deviation with increasing self-preference σ. Furthermore, the average opinion plots show that the overall group average opinion doesn't change by large amounts over the courses of the simulations, with the percent change fluctuating randomly between $\sim \pm 10\%$. The standard deviations, on the other hand, all show a systematic increase after running the model, with a persistent positive percent change often on the order or $\sim 30\%$.

The results of the sensitivity analysis imply that whilst the global group opinion is relatively static, local changes drive equal and opposite spreading of agent opinions such that the global average is constant but the standard deviation increases. Inspection of the model results indeed showed this effect, and we present here two representative examples: in Fig. 5 the model parameters were $\gamma = 0.1$ (grid units)$^{-1}$ and $\sigma = 0.5$ whilst in Fig. 6 the model parameters were $\gamma = 0.1$ (grid units)$^{-1}$ and $\sigma = 0.9$. The top of both figures shows frequency histograms of initial and final agent opinion, whilst the bottom shows explicit initial and final spatial distributions of opinions. As can be seen in the histograms, there is emergence of opinion polarisation from approximately symmetric unimodal initial opinion histograms, whilst examining the spatial distribution shows emergence of local contiguous extremist clusters from randomly mixed opinion distributions, with the size of these clusters slightly increasing as group self-preference σ increases from 0.5 (no preference) to 0.9 (strong preference).

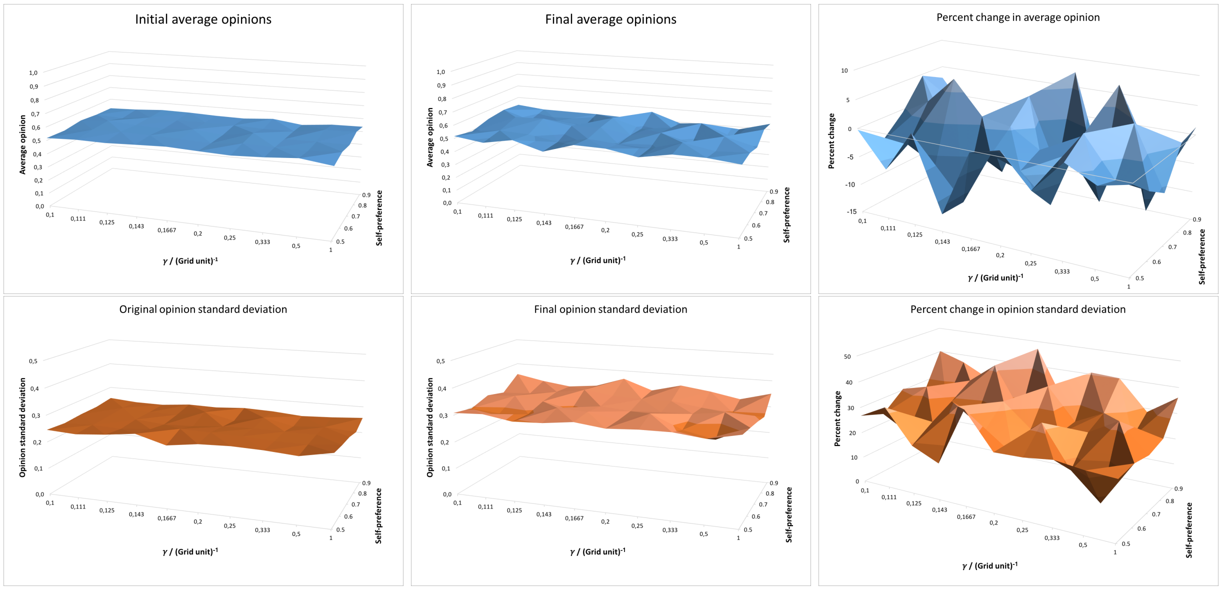
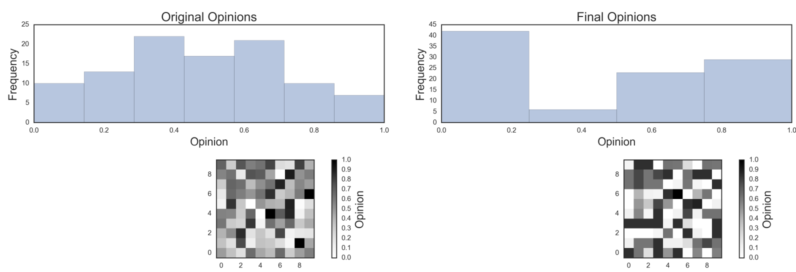


Fig. 4: Sensitivity of model to γ (x-axis) and σ (y-axis). Top row (from left): initial average agent opinion, final average opinion, percent change in average opinion. Bottom row (from left): initial agent opinion standard deviation, final standard deviation, percent change in standard deviation. Each point is an average of five independent model runs.



(a) Initial state.

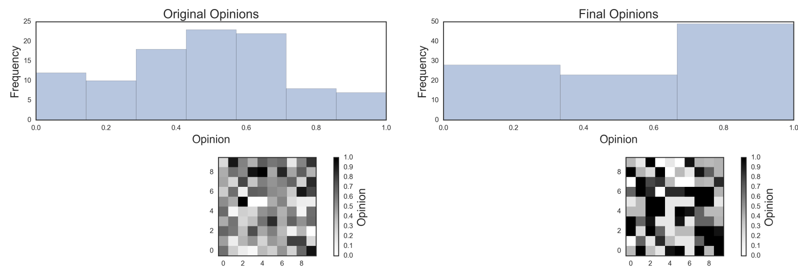
(b) Final state.

Fig. 5: Results from a simulation with $\gamma = 0.1$ (grid units)$^{-1}$ and $\sigma = 0.5$: (a) initial opinion distributions (b) final opinion distributions.

5 Discussion

By running this model with stochastic, chemistry-inspired transition probabilities, we are able to generate results that overall are consistent with the outcomes of Brexit.

First, generating winning probability timecourses demonstrate the chaos/randomness of the model by converging to different states with the same model parameters, as well as stressing the noisy effect of probabilistic voting behaviour given demographics, consistent with the inability of polls to predict the Brexit outcome, which our model suggest may have been due to not realising the stochasticity of real voters or improper assumptions of equal voting probability across all demographic groups.



(a) Initial state. (b) Final state.

Fig. 6: Results from a simulation with $\gamma = 0.1$ (grid units)$^{-1}$ and $\sigma = 0.9$: (a) initial opinion distributions (b) final opinion distributions.

Second, our sensitivity analysis shows that the model is overall not exquisitely sensitive to the exact model parameter γ and only slightly so to σ, suggesting instead that just the exponential interaction mechanism is enough on its own to generate the observed results. The results themselves show a global moderate opinion, though a medium average opinion was shown by examining specific simulation results to be a result of bimodal extreme opinion histograms diagnostic of polarisation, and examination of spatial opinion distributions showed emergence of local extreme ideological clusters. These observations are consistent with the ultimately close Brexit vote outcome despite very strongly polarised and clustered opinions.

It is important to realise the limitations of our approach – perhaps most significantly the lack of a full factorial analysis due to finite available computing power. In other words, we have not examined here how the model depends on other parameters, such as grid size, standard deviation of initial opinions, step size of inducibility reduction upon unsuccessful interactions, etc. Future directions of this work include such higher-dimensional sensitivity analysis.

6 Conclusion

Despite the lack of explicit extremists, we have developed here a grid-based, opinion dynamics ABM that results in polarisation and opinion extremism, due in large part to a chemistry-based interaction mechanism and slightly to preferential self-group exchanges. Our model predicts that an initially moderate, well-mixed society can converge to a strongly polarised, locally-clustered society in social-group-space, producing very close results in democratic decision-making events. Furthermore, we show that stochastic effects, as well as non-unity voting probabilities based on demographics, render outcome predictions difficult to generate and interpret, and almost impossible to produce for close results.

These results are in accordance with recent political events such as the US 2016 presidential election or the UK 2016 EU referendum, and thus heighten our understanding of group dynamics in cases in which social division is important. Our results suggest that moderate opinions lead to extreme polarisation simply through interactions in which discourse quickly becomes more difficult as opinion similarity decreases, and that social group preference increases this effect, leading to local clustering of extreme ideologies.

References

1. (2016) EU referendum: The result in maps and charts. British Broadcasting Corporation. [Online]. Available: http://www.bbc.com/news/uk-politics-36616028
2. (2016) Yougov on the dall poll survey results. YouGov. [Online]. Available: https://d25d2506sfb94s.cloudfront.net/cumulus_uploads/document/640yx5m0rx/On_the_Day_FINAL_poll_forwebsite.pdf
3. (2016) Pound plunges after Leave vote. British Broadcasting Company. [Online]. Available: http://www.bbc.com/news/business-36611512
4. (2016) Race and religious hate crimes rose 41% after EU vote. British Broadcasting Company. [Online]. Available: http://www.bbc.com/news/uk-politics-37640982
5. N. McCarthy. (2016) Report: Trump's Election Led To A Surge In Hate Crime. Forbes. [Online]. Available: http://www.forbes.com/sites/niallmccarthy/2016/11/30/report-trumps-election-led-to-a-surge-in-hate-crime-infographic/
6. (2016) The dollars strength is a problem for the world. The Economist. [Online]. Available: http://www.economist.com/news/finance-and-economics/21711037-what-recent-strength-dollar-means-global-economy-dollars
7. A. Boult. (2016) Donald Trump calls himself Mr Brexit confusion and ridicule ensue. The Telegraph. [Online]. Available: http://www.telegraph.co.uk/news/2016/08/18/donald-trump-calls-himself-mr-brexit–confusion-and-ridicule-ens/
8. M. Karnitschnig. (2016) Trump dashes Europes hopes of Atlanticist revival. Politico. [Online]. Available: http://www.politico.eu/article/donald-trump-dashes-europes-hopes-of-atlanticist-revival-frenzy-le-pen/
9. U. Krause, "A discrete nonlinear and non-autonomous model of consensus formation," in *Communications in Difference Equations*. Gordon and Breach Pub, 2000, pp. 227–236.
10. J. C. Dittmer, "Consensus formation under bounded confidence," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 7, pp. 4615–4621, 2001.
11. G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, "Mixing beliefs among interacting agents," *Advances in Complex Systems*, vol. 3, no. 01n04, pp. 87–98, 2000.
12. G. Deffuant, F. Amblard, G. Weisbuch, and T. Faure, "How can extremism prevail? a study based on the relative agreement interaction model," *Journal of Artificial Societies and Social Simulation*, vol. 5, no. 4, 2002.
13. (2011) 2011 Census: KS102EW Age structure, local authorities in England and Wales. Office for National Statistics. [Online]. Available: http://webarchive.nationalarchives.gov.uk/20160105160709/
14. (2013) Graduates in the UK Labour Market 2013. Office for National Statistics. [Online]. Available: http://www.ons.gov.uk/ons/dcp171776_337841.pdf
15. T. Helm. (2016) EU referendum: youth turnout almost twice as high as first thought. The Guardian. [Online]. Available: https://www.theguardian.com/politics/2016/jul/09/young-people-referendum-turnout-brexit-twice-as-high
16. P. Atkins and J. De Paula, *Atkins Physical Chemistry*. New York: Oxford University Press, 2006.

Appendix A Model parameters

Parameter	Description	Value(s)
L / Grid units	Grid size	10
f_a	Fraction of agents without university degree	0.617
f_A	Fraction of agents younger than 50	0.55825
p(vote|A)	Voting likelihood of the young	0.65
p(vote|B)	Voting likelihood of the old	0.7995
σ	Group self-preference	$[0.5, 0.6, 0.7, 0.8, 0.9]$
γ / (Grid units)$^{-1}$	Search probability parameter	$[0.1, 0.111, 0.125, 0.143, 0.1667, 0.2, 0.25, 0.333, 0.5, 1]$
$\langle o \rangle_A$	Average opinion of the young (for initialisation)	0.5998
$\langle o \rangle_A$	Average opinion of the old (for initialisation)	0.4169
$\langle o \rangle_a$	Average opinion of non-degree holders (for initialisation)	0.456
$\langle o \rangle_b$	Average opinion of degree holders (for initialisation)	0.710

Table 1: Parameters used for running the model.

Appendix B Search probability radial distribution function

Define a 2D polar coordinate system on the model grid centred about agent i, with agent j at coordinate $\boldsymbol{r} = (r_{ij}, \theta)$. Assuming independent probability densities in the radial and polar directions, the overall search probability distribution function can be written as:

$$p(r, \theta) = p_r(r)p_\theta(\theta) \tag{2}$$

Allowing for the radial bit to be exponentially distributed such that $p_r(r) = N_r \exp(-\gamma r)$ and for the polar bit to be uniformally distributed such that $p_\theta(\theta) = N_\theta$, for N_r and N_θ normalisation constants, then gives:

$$p(r, \theta) = N_r N_\theta \exp(-\gamma r) \tag{3}$$

Normalisation requires that:

$$\begin{aligned} 1 &= \int_V dV p(r, \theta) && (4) \\ &= \int_0^{2\pi} d\theta N_\theta \int_0^{L/2} r dr N_r \exp(-\gamma r) && (5) \\ &= \int_0^{L/2} dr P_r(r), && (6) \end{aligned}$$

where in the last line we have introduced the effective radial distribution $P_r(r) = r p_r(r) = N_r r \exp(-\gamma r)$. Integrating gives the normalisation constant, and yields a final expression for $P_r(r)$:

$$P_r(r) = \frac{\gamma^2 r \exp(-\gamma r)}{1 - \exp(-\frac{\gamma L}{2})(1 + \frac{\gamma L}{2})} \tag{7}$$